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Map for Simultaneous Measurements for a Quantum Logic

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In this paper we will study a function of simultaneous measurements for quantum events (s-map) which will be compared with the conditional states on an orthomodular lattice as a basic structure for quantum logic. We will show the connection between s-map and a conditional state. On the basis of the Rényi approach to the conditioning, conditional states, and the independence of events with respect to a state are discussed. Observe that their relation of independence of events is not more symmetric contrary to the standard probabilistic case. Some illustrative examples are included.

KEY WORDS: simultaneous measurements; quantum logic.

1. INTRODUCTION

Conditional probability plays a basic role in the classical probability theory. Some of the most important areas of the theory such as martingales, stochastic processes rely heavily on this concept. Conditional probabilities on a classical measurable space are studied in several different ways, but result in equivalent theories. The classical probability theory does not describe the causality model.

The situation changes when nonstandard spaces are considered. For example, it is well known that the set of random events in quantum mechanics experiments is a more general structure than Boolean algebra. In the quantum logic approach the set of random events is assumed to be an orthomodular lattice (OML) L. Such model we can find not only in the quantum theory, but also for example, in economics, biology, etc. We will show such a simple situation in Example 1.

In this paper we will study a conditional state on an OML using Renyi's approach (or Bayesian principle). This approach helps us to define independence of events and differently from the situation in the classical theory of probability, if an event a is independent of an event b, then the event b can be dependent on the event a (problem of causality) (Nánásiová, 1998, 2001). We will show that we can define an s-map (function for simultaneous measurements on an OML).

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It can be shown that if we have the conditional state we can define the s-map and conversely. By using the s-map we can introduce joint distribution also for noncompatible observables on an OML. Moreover, if x is an observable on L and B is Boolean subalgebra of L, we can construct an observable $z = E(x|B)$, which is a version of conditional expectation of x but it need not to be necessarily compatible with x.

Example 1. Assume that there are four objects (A, U), (A, V), (C, U), (C, V) under experimental observations, but because of the nature of experimental device we are able to identify only one constituent of the pair. Thus, the possible outcomes of our experiment are A, C, U, V, and if the outcome is (say) A then we do not know whether it comes from the pair (A, U) or from (A, V). In other words we can always observe only one characteristic feature of each object:

$$A = \Pi_1(A, U) = \Pi_1(A, V) \qquad C = \Pi_1(C, U) = \Pi_1(C, V)$$
$$U = \Pi_2(A, U) = \Pi_2(C, U) \qquad V = \Pi_2(A, V) = \Pi_2(C, V)$$

where Π_i, $i = 1, 2$ present some "state" of our system. In such situation, for example, we ask about the probability of A if property U has been detected; equivalently we ask about the value of $P(A|U)$.

2. A CONDITIONAL STATE ON AN OML

In this part we introduce the notions as an OML, a state, a conditional state, and their basic properties.

Definition 1.1. Let L be a nonempty set endowed with a partial ordering $\leq$. Let there exist the greatest element (1) and the smallest element (0). Let there be defined the operations supremum ($\vee$), infimum $\wedge$ (the lattice operations) and a map $\perp : L \to L$ with the following properties:

(i) For any $\{a_n\}_{n\in\mathcal{A}} \in L$, where $\mathcal{A} \subset \mathcal{N}$ are finite
$$\bigvee_{n\in\mathcal{A}} a_n, \quad \bigwedge_{n\in\mathcal{A}} a_n \in L.$$
(ii) For any $a \in L (a^\perp)^\perp = a$.
(iii) If $a \in L$, then $a \vee a^\perp = 1$.
(iv) If $a, b \in L$ such that $a \leq b$, then $b^\perp \leq a^\perp$.
(v) If $a, b \in L$ such that $a \leq b$ then $b = a \vee (a^\perp \wedge b)$ (orthomodular law).

Then $(L, 0, 1, \vee, \wedge, \perp)$ is called the orthomodular lattlice (briefly OML).

Let L be OML. Then elements $a, b \in L$ will be called:

- orthogonal ($a \perp b$) iff $a \leq b^{\perp}$;
- compatible ($a \leftrightarrow b$) iff there exist mutually orthogonal elements a_1, b_1, $c \in L$ such that

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c.$$

If $a_i \in L$ for any $i = 1, 2, 3, \ldots$ and $b \in L$ is such, that $b \leftrightarrow a_i$ for all i, then $b \leftrightarrow \bigvee_{i=1}^{n} a_i$ and (Dvurečenskij and Pulmannová, 2000; Pták and Pulmannová, 1991; Varadarajan, 1968)

$$b \wedge \left(\bigvee_{i=1}^{\infty} a_i \right) = \bigvee_{i=1}^{\infty} (a_i \wedge b)$$

A subset $L_0 \subseteq L$ is a sublogic of L if for any $a \in L_0$ we have $a^{\perp} \in L_0$ and for any $a, b \in L_0$ $a \vee b \in L_0$.

Definition 1.2. A map $m : L \to R$ such that

(i) $m(0) = 0$ and $m(1) = 1$.
(ii) If $a \perp b$ then $m(a \vee b) = m(a) + m(b)$

is called a state on L. If we have orthomodular σ-lattice and m is σ-additive function, then m will be called a σ-state.

Definition 1.3. (Nánásiová, 2001). Let L be an OML. A subset $L_c \subset L - \{0\}$ is called a conditional system (CS) in L (σ-CS in L) if the following conditions hold:

- If $a, b \in L_c$, then $a \vee b \in L_c$. (If $a_n \in L_c$, for $n = 1, 2, \ldots$, then $\bigvee_n a_n \in L_c$.)
- If $a, b \in L_c$ and $a < b$, then $a^{\perp} \wedge b \in L_c$.

Let $A \subset L$. Then $L_c(A)$ is the smallest CS (σ-CS), which contains the set A.

Definition 1.4. (Nánásiová, 2001). Let L be an OML and L_c be a σ-CS in L. Let $f : L \times L_c \to [0, 1]$. If the function f fulfills the following conditions:

(C1) for each $a \in L_0$ $f(., a)$ is a state on L;
(C2) for each $a \in L_0$ $f(a, a) = 1$;
(C3) if $\{a_n\}_{n \in \mathcal{A}} \in L_0$, where $\mathcal{A} \subset \mathcal{N}$ and a_n are mutually orthogonal, then for each $b \in L$

$$f\left(b, \bigvee_{n \in \mathcal{A}} a_n\right) = \sum_{n \in \mathcal{A}} f\left(a_n, \bigvee_{n \in \mathcal{A}} a_n\right) f(b, a_n);$$

then it is called conditional state.

Proposition 1.1. *(Nánásiová, 2001). Let L be an OML. Let $\{a_i\}_{i=1}^n \in L$, $n \in N$ where $a_i \perp a_j$ for $i \neq j$. If for any i there exists a state α_i, such that $\alpha_i(a_i) = 1$, then there exists σ-CS such that for any $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, where $k_i \in [0; 1]$ for $i \in \{1, 2, \ldots, n\}$ with the property $\sum_{i=1}^n k_i = 1$, there exists a conditional state*

$$f_{\mathbf{k}} : L \times L_c \to [0; 1],$$

such that

1. *for any i and each $d \in L$ $f_{\mathbf{k}}(d, a_i) = \alpha_i(d)$;*
2. *for each a_i*

$$f_{\mathbf{k}}\left(a_i \bigvee_{i=1}^n a_i\right) = k_i;$$

Definition 1.5. (Nánásiová, 2001). Let L be an OML and f be a conditional state. Let $b \in L$, $a, c \in L_c$ such that $f(c, a) = 1$. Then b is independent of a with respect to the state $f(., c)(b \asymp_{f(.,c)} a)$ if $f(b, c) = f(b, a)$.

The classical definition of independency of a probability space $(\Omega, \mathcal{B}, \mathcal{P})$ is a special case of this definition, because

$$P(A|B) = P(A|\Omega) \text{ if and only if } P(A \cap B|\Omega) = P(A|\Omega)P(B|\Omega).$$

If L_c be CS and $f : L \times L_c \to [0, 1]$ is a conditional state, then (Nánásiová, 2001)

(i) Let $a^\perp, a, c \in L_c$, $b \in L$ and $f(c, a) = f(c, a^\perp) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $b \asymp_{f(.,c)} a^\perp$.
(ii) Let $a, c \in L_c$, $b \in L$ and $f(c, a) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $b \frac{\perp}{f}(., c) a$.
(iii) Let $a, c, b \in L_c$, $b \leftrightarrow a$ and $f(c, a) = f(c, b) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $a \asymp_{f(.,c)} b$.

3. FUNCTION FOR SIMULTANEOUS MEASUREMENT (*s*-MAP)

Definition 2.1. Let L be an OML. The map $p : L \times L \to [0, 1]$ will be called s-map if the following conditions hold:

(s1) $p(1, 1) = 1$;
(s2) if $a \perp b$, then $p(a, b) = 0$;
(s3) if $a \perp b$, then for any $c \in L$,

$$p(a \vee b, c) = p(a, c) + p(b, c)$$

$$p(c, a \vee b) = p(c, a) + p(c, b)$$

Proposition 2.1. *Let L be an OML and let p be a s-map. Let $a, b, c \in L$, then*

1. *if $a \leftrightarrow b$, then $p(a, b) = p(a \wedge b, a \wedge b) = p(b, a)$;*
2. *if $a \leq b$, then $p(a, b) = p(a, a)$;*
3. *if $a \leq b$, then $p(a, c) \leq p(b, c)$;*
4. *$p(a, b) \leq p(b, b)$;*
5. *if $\nu(b) = p(b, b)$, then ν is a state on L.*

Proof:

(1) If $a \leftrightarrow b$, then $a = (a \wedge b) \vee (a \wedge b^{\perp})$ and $b = (b \wedge a) \vee (b \wedge a^{\perp})$. Hence

$$\begin{aligned} p(a, b) &= p((a \wedge b) \vee (a \wedge b^{\perp}), b) \\ &= p(a \wedge b, b) + p(a \wedge b^{\perp}, b) = p(a \wedge b, b). \end{aligned}$$

Analogously

$$\begin{aligned} p(a \wedge b, b) &= p(a \wedge b, (b \wedge a) \vee (b \wedge a^{\perp})) \\ &= p(b \wedge a, b \wedge a) + p(b \wedge a, b \wedge a^{\perp}) = p(b \wedge a, b \wedge a). \end{aligned}$$

Hence

$$p(a, b) = p(a \wedge b, a \wedge b).$$

(2) If $a \leq b$, then $a \leftrightarrow b$. Hence

$$p(a, b) = p(a, a \wedge b) = p(a, a).$$

(3) If $a \leq b$, then $b = a \vee (a^{\perp} \wedge b)$. Hence

$$\begin{aligned} p(b, c) &= p(a \vee (a^{\perp} \wedge b), c) \\ &= p(a, c) + p(a^{\perp} \wedge b, a) p(a, c) \end{aligned}$$

(4) From (3) and (2) it follows

$$p(b, b) = p(1, b) p(a, b).$$

Hence we get

$$p(b, b) p(a, b) \quad \text{for each} \quad a, b \in L.$$

(5) Let $\nu : L \to [0, 1]$, such that $\nu(b) = p(b, b)$. Then

$$\nu(0) = p(0, 0) = 0.$$

Let $a \perp b$, then

$$\begin{aligned} \nu(a \vee b) &= p(a \vee b, a \vee b) = p(a, a \vee b) + p(b, a \vee b) \\ &= p(a, a) + p(a, b) + p(b, a) + p(b, b) \\ &= p(a, a) + p(b, b) = \nu(a) + \nu(b). \end{aligned}$$

From the definition we have that $\nu(1) = p(1, 1) = 1$. From this it follows that ν is a state on L. □

Proposition 2.2. *Let L be an OML, let there be an s-map p. Then there exists a conditional state f_p, such that*

$$p(a, b) = f_p(a, b) f_p(b, 1).$$

Let L be an OML and let $L_c = L - \{0\}$. If $f : L \times L_c \to [0, 1]$ is a conditional state, then there exists an s-map $p_f : L \times L \to [0, 1]$.

Proof: Let p be an s-map. Let $L_c = \{b \in L; p(b, b) \neq 0\}$. Let $f_p : L \times L_c \to R$ such that

$$f_p(., b) = \frac{p(., b)}{p(b, b)}.$$

From the Proposition 2.1 (3) it follows that for any $a \in L$ and $b \in L_c$ $f_p(a, b) \in [0, 1]$. Moreover

$$f_p(0, b) = 0 \quad \text{and} \quad f_p(1, b) = \frac{p(1, b)}{p(b, b)} = \frac{p(b, b)}{p(b, b)} = 1$$

and also $f_p(b, b) = 1$. Let $c, a \in L$ and let $a \perp c$. Then

$$f_p(a \vee c, b) = \frac{p(a \vee c, b)}{p(b, b)} = \frac{p(a, b) + p(c, b)}{p(b, b)} = f_p(a, b) + f_p(c, b).$$

It means that for any $b \in L_c$ is $f_p(., b)$ a state on L.

Let $b_i \in L_c$, $i = 1, 2, \ldots, n$ be mutually orthogonal elements. Then for any $a \in L$

$$\begin{aligned} f_p\left(a, \bigvee_{i=1}^{n} b_i\right) = \frac{p(a, \vee_i b_i)}{p(\vee_i b_i, \vee_i b_i)} &= \sum_{i=1}^{n} \frac{p(a, b_i)}{p(\vee_i b_i, \vee_i b_i)} \\ &= \sum_{i=1}^{n} \frac{p(b_i, \vee_i b_i)}{p(\vee_i b_i, \vee_i b_i)} \frac{p(a, b_i)}{p(b_i, \vee_i b_i)} \\ &= \sum_{i=1}^{n} \frac{p(b_i, \vee_i b_i)}{p(\vee_i b_i, \vee_i b_i)} \frac{p(a, b_i)}{p(b_i, b_i)} \\ &= \sum_{i=1}^{n} f_p(b_i, \vee_i b_i) f(a, b_i). \end{aligned}$$

From this it follows that f_p is the conditional state.

Now we can compute

$$f_p(a,b)\,f_p(b,1) = \frac{p(a,b)}{p(b,b)}\frac{p(b,1)}{p(1,1)}.$$

From the properties of s-map we have $p(b,1) = p(b,b)$ and $p(1,1) = 1$. Hence $f_p(a,b)\,f_p(b,1) = p(a,b)$.

Let f be a conditional state and let $L_0 = \{b \in L_c;\ f(b,1) \neq 0\}$. Let

$$p_f : L \times L \to [0,1]$$

be defined in the following way:

$$p_f(a,b) = \begin{cases} f(a,b)\,f(b,1), & b \in L_0 \\ 0, & b \notin L_0 \end{cases}$$

(s1) Because $1 \in L_0$ and f is a conditional state, then

$$p_f(1,1) = f(1,1)\,f(1,1) = 1.$$

(s2) Let $a, b \in L$ and $a \perp b$. If $b \in L_0$, then $p_f(a,b) = f(a,b)\,f(b,1)$. Because $a \leq b^{\perp}$, then $f(a,b) = 0$. Hence $p_f(a,b) = 0$. If $b \notin L_0$, then $p_f(a,b) = 0$. Hence for $a \perp b p_f(a,b) = 0$.

(s3) Let $a, b, c \in L$, $a \perp b$. We have to show that

$$p_f(a \vee b, c) = p_f(a,c) + p_f(b,c) \tag{1}$$

and

$$p_f(c, a \vee b) = p_f(c,a) + p_f(c,b). \tag{2}$$

(1) If $c \in L_0$, then

$$\begin{aligned} p_f(a \vee b, c) &= f(a \vee b, c)\,f(c,1) \\ &= f(a,c)\,f(c,1) + f(b,c)\,f(c,1) \\ &= p_f(a,c) + p_f(a,c). \end{aligned}$$

If $c \notin L_0$, then $p_f(a \vee b, c) = p_f(a,c) = p_f(b,c) = 0$. Hence

$$p_f(a \vee b, c) = p_f(a,c) + p_f(b,c).$$

(2) In this case we have to verify for (b) the following three situations: (i) $a, b \in L_0$; (ii) $a \in L_0$, $b \notin L_0$; (iii) $a, b \notin L_0$.

(i) If $a, b \in L_0$, then

$$\begin{aligned} p_f(c, a \vee b) &= f(c, a \vee b)\,f(a \vee b, 1) \\ &= (f(a, a \vee b)\,f(c,a) + f(b, a \vee b)\,f(c,b))\,f(a \vee b, 1) \\ &= f(c,a)\,f(a, a \vee b)\,f(a \vee b, 1) \\ &\quad + f(c,b)\,f(b, a \vee b)\,f(a \vee b, 1). \end{aligned}$$

From the definition of the function f we get

$$f(a,1) = f(a, a\vee b)f(a\vee b,1) + f(a,(a\vee b)^{\perp})f((a\vee b)^{\perp},1)$$
$$= f(a,a\vee b)f(a\vee b,1) + 0.$$

Also

$$f(b,a\vee b)f(a\vee b,1) = f(b,1).$$

Then

$$\begin{aligned} p_f(c,a\vee b) &= f(c,a)f(a,a\vee b)f(a\vee b,1) \\ &\quad + f(c,b)f(b,a\vee b)f(a\vee b,1) \\ &= f(c,a)f(a,1) + f(c,b)f(b,1) \\ &= p_f(c,a) + p_f(c,b). \end{aligned}$$

(ii) If $a \in L_0$ and $b \notin L_0$ and $a\vee b \in L_0$, then from the definition of a map p_f it follows $p_f(c,b) = 0$. From this it follows that it is enough to show

$$p_f(c,a\vee b) = p_f(c,a).$$

But

$$p_f(c,a\vee b) = f(c,a\vee b)f(a\vee b,1)$$

and

$$p_f(c,a) = f(c,a)f(a,1).$$

Because $f(b,1) = 0$, then

$$f(a\vee b,1) = f(a,1) + f(b,1) = f(a,1).$$

On the other hand

$$\begin{aligned} 0 = f(b,1) &= f(a\vee b,1)f(b,a\vee b) \\ &\quad + f((a\vee b)^{\perp},1)f(b,(a\vee b)^{\perp}). \end{aligned}$$

Because $f(b,(a\vee b)^{\perp}) = 0$, then we have

$$0 = f(a\vee b,1)f(b,a\vee b).$$

But $f(a\vee b,1) \neq 0$ and hence

$$f(b,a\vee b) = 0$$

and so

$$1 = f(a\vee b,a\vee b) = f(a,a\vee b) + f(b,a\vee b) = f(a,a\vee b).$$

Therefore

$$f(c, a \vee b) = f(a, a \vee b) f(c, a) + f(b, a \vee b) f(c, b) = f(c, a).$$

Hence

$$\begin{aligned} p_f(c, a \vee b) &= f(c, a \vee b) f(a \vee b, 1) \\ &= f(c, a) f(a, 1) = p_f(c, a). \end{aligned}$$

(iii) If $a, b \notin L_0$, then $f(a, 1) = f(b, 1) = 0$. From this it follows that $f(a \vee b, 1) = 0$ and so $a \vee b \notin L_0$. Hence for any $c \in L$

$$0 = p_f(c, a \vee b) = p_f(c, a) + p_f(c, b).$$

Therefore p_f is s-map.

□

Proposition 2.3. *Let L be an OML.*

(a) *If f is a conditional state, then $b \asymp_{f(.,1)} a$ iff $p_f(b, a) = p_f(a, a) p_f(b, b)$, where p_f is the s-map generated by f.*
(b) *Let p be an s-map. Then $b \asymp_{fp(.,1)} a$ iff $p(b, a) = p(a, a)p(b, b)$, where f_p is the conditional state generated by the s-map p.*

Proof:

(a) Let $b \asymp_{f(.,1)} a$. It means that $f(b, a) = f(b, 1)$. Let $f(b, 1) \neq 0$ and $f(a, 1) \neq 0$. From the previous proposition we have that

$$p_f(b, a) = f(b, a) f(a, 1) = f(b, 1) f(a, 1).$$

But

$$p_f(d, d) = f(d, d) f(d, 1) = f(d, 1)$$

and hence

$$p_f(b, a) = p_f(b, b) p_f(a, a).$$

Let $f(b, 1) = 0$ and $f(a, 1) \neq 0$. From this it follows that $p_f(b, b) = f(b, 1) = 0$. On the other hand

$$f(b, 1) = f(a, 1) f(b, a) + f(a^{\perp}, 1) f(b, a^{\perp}) = 0.$$

Therefore $f(b, a) = 0$ and hence $p_f(b, a) = 0$. It means that in this case $p_f(b, a) = p_f(b, b) p_f(a, a)$.

Let $f(b, 1) = f(a, 1) = 0$. From this it follows that $f(a, 1) = p_f(a, a) = 0 = p_f(b, b)$ and so $p_f(a, a) p_f(b, b) = 0$. On the other hand

$p_f(b,a) = f(b,a)f(a,1) = 0$. It means

$$b \asymp_{f(.,1)} a \quad \text{implies} \quad p_f(b,a) = p_f(a,a)p_f(b,b). \tag{3}$$

If $p_f(b,a) = p_f(a,a)p_f(b,b)$, then $p_f(b,a) = f(a,1)f(b,1)$. It means that

$$p_f(b,a) = f(b,a)f(a,1) = f(b,1)f(a,1).$$

From this it follows

$$f(b,1) = f(b,a),$$

and so

$$b \asymp_{f(.,1)} a.$$

(b) Let p be an s-map and $L_c = \{d \in L; p(d,d) \neq 0\}$. Let $f_p : L \times L_c \to [0;1]$ be the conditional state defined

$$f_p(b,a) = \frac{p(b,a)}{p(a,a)}.$$

Let $b \asymp_{f_p(.,1)} a$. It means that $f_p(b,a) = f_p(b,1)$. Hence

$$f_p(b,a) = \frac{p(b,a)}{p(a,a)}$$

and

$$f_p(b,1) = \frac{p(b,1)}{p(1,1)} = p(b,b).$$

Hence

$$\frac{p(b,a)}{p(a,a)} = p(b,b)$$

and so

$$p(b,a) = p(a,a)p(b,b).$$

On the other hand, if $p(a,b) = p(a,a)p(b,b)$, then

$$\begin{aligned} f_p(b,a) &= \frac{p(b,a)}{p(a,a)} = \frac{p(a,a)p(b,b)}{p(a,a)} \\ &= p(b,b) = p(b,1) \\ &= \frac{p(b,1)}{p(1,1)} = f_p(b,1). \end{aligned}$$

It means $b \asymp_{f_p(.,1)} a$.

□

Example 2.1. Let $L = \{a, a^\perp, b, b^\perp, 0, 1\}$. It is clear that L is an OML. Let $f(s, t)$ is defined by the following way:

s/t	a	$a^\perp$	b	$b^\perp$	1
a	1	0	0.4	0.4	0.4
$a^\perp$	0	1	0.6	0.6	0.6
b	0.2	11/30	1	0	0.3
$b^\perp$	0.8	19/30	0	1	0.7

From f we can compute $p_f(s, t)$. Then we get

s/t	a	$a^\perp$	b	$b^\perp$
a	0.4	0	0.12	0.28
$a^\perp$	0	0.6	0.18	0.42
b	0.08	0.22	0.3	0
$b^\perp$	0.32	0.38	0	0.7

We can see that $p_f(a, b) = p_f(a, a)p_f(b, b)$, but $p_f(b, a) \neq p_f(b, b)p_f(a, a)$.

4. ON OBSERVABLES

Let $\mathcal{B}(\mathcal{R})$ be σ-algebra of Borel sets. A σ-homomorphism $x : \mathcal{B}(\mathcal{R}) \to \mathcal{L}$ is called an observable on L. If x is an observable, then $R(x) := \{x(E); E \in \mathcal{F}\}$ is called range of the observable x. It is clear that $R(x)$ is Boolean σ-algebra [Var]. Let us denote $\nu(b) = p(b, b)$ for $b \in L$.

Definition 3.1. Let L be a σ-OML and $p : L \times L \to [0; 1]$ be an s-map. Let x, y be some observables on L. Then a map $p_{x,y} : \mathcal{B}(\mathcal{R}) \times \mathcal{B}(\mathcal{R}) \to [\prime, \infty]$, such that

$$p_{x,y}(E, F) = p(x(E), y(F)),$$

is called a joint distribution for the observables x and y.

If $F_{x,y}(r, s) = p(x(-\infty, r), y(-\infty, s))$, then the function $F_{x,y}$ is the distribution function of the observables x, y. It is clear that for $r_1 \leq r_1$, then $F_{x,y}(r_1, 8) \leq F_{x,y}(r_2, s)$.

If x is an observable on L and m is a state on L, then $m_x(E)$, $E \in \mathcal{B}(\mathcal{R})$ is probability distribution for x and

$$m(x) = \int_R \lambda m_x\,(d\lambda)$$

is called the expectation of x in the state m, if the integral on the right side exists.

Definition 3.2. Let x be an observable on L and B be a Boolean subalgebra of L and f be conditional state on L such that $L_c = L - \{0\}$. Then the observable z will be called a conditional expectation of x with respect to B in the state $f(., 1)$ iff for any $b \in B - \{0\}$

$$f(x, b) = f(z, b).$$

We will denote $z := E_f(x|B)$.

It is clear that if L be a Boolean algebra, then $E_f(x|B)$ is known the conditional expectation. The expectations of x in the state m have been studied in many papers Dvurečenskij and Pulmannová, 2000; Gudder, 1965, 1966, 1967, 1968, 1969, 1984; Gudder and Mullikin, 1984; Gudder and Piron, 1971; Nánásiová, 1987a, 1993a,b; Nánásiová and Pulmannová, 1985; Pták and Pulmannová, 1991), etc. In the end we show that such conditional expectation can exist on L.

Example 3.1. Let L be the same as in Example 2.1. We have the set

$$\{f(., a),\ f(., a^{\perp}),\ f(., b),\ f(., b^{\perp}),\ f(., 1)\}$$

of states and $B_d = \{0, 1, d, d^{\perp}\}$, where $d \in L$. Let x, z be observales on L such that $R(x) = B_a$. and $R(z) = B_b$. It is easy to see, that x is not compatible with z. Let

$$x(r_1) = a \quad x(r_2) = a^{\perp}$$
$$z(s_1) = b \quad z(s_2) = b^{\perp}$$

for $r_1, r_2, s_1, s_2 \in R$.

If $z = E_f(x|B)$, then

$$f(x, b) = f(z, b), \qquad f(x, b^{\perp}) = f(z, b^{\perp}), \qquad f(x, 1) = f(z, 1).$$

From the definition of the expectation of an observable we have

$$\begin{aligned} f(x, 1) &= r_1 f(a, 1) + r_2 f(a^{\perp}, 1) = f(z, 1) \\ &= s_1 f(b, 1) + s_2 f(b^{\perp}, 1), \\ f(x, b) &= r_1 f(a, b) + r_2 f(a^{\perp}, b) = f(z, b) \\ &= s_1 f(b, b) + s_2 f(b^{\perp}, b) = s_1, \\ f(x, b^{\perp}) &= r_1 f(a, b^{\perp}) + r_2 f(a^{\perp}, b^{\perp}) = f(z, b^{\perp}) \\ &= s_1 f(b, b^{\perp}) + s_2 f(b^{\perp}, b^{\perp}) = s_2. \end{aligned}$$

Let $s_1 \neq s_2$. If we put

$$s_1 = r_1 f(a, b) + r_2 f(a^\perp, b)$$

and

$$s_2 = r_1 f(a, b^\perp) + r_2 f(a^\perp, b^\perp),$$

then

$$\begin{aligned} f(z, 1) &= s_1 f(b, 1) + s_2 f(b^\perp, 1) \\ &= \left[r_1 f(a, b) + r_2 f(a^\perp, b)\right] f(b, 1) \\ &\quad + \left[r_1 f(a, b^\perp) + r_2 f(a^\perp, b^\perp)\right] f(b^\perp, 1) \\ &= r_1[f(a, b) f(b, 1) + f(a, b^\perp) f(b^\perp, 1)] \\ &\quad + r_2[f(a^\perp, b) f(b, 1) + f(a^\perp, b^\perp) f(b^\perp, 1)] \\ &= r_1 f(a, 1) + r_2 f(a^\perp, 1) = f(x, 1). \end{aligned}$$

From this it follows that $z = E_f(x|B)$.

If $a \asymp_{f(.,1)} b$, then $f(a, b) = f(a, 1) = f(a, b^\perp)$. From the definition of the expectation of an observable we have

$$\begin{aligned} f(x, b) &= r_1 f(a, 1) + r_2 f(a^\perp, 1) = f(z, b) = f(z, 1) = s, \\ f(x, b^\perp) &= r_1 f(a, 1) + r_2 f(a^\perp, 1) = f(z, b^\perp) = f(z, 1) = s \\ f(x, 1) &= r_1 f(a, 1) + r_2 f(a^\perp, 1) = f(z, 1) \\ &= s_1 f(b, 1) + s_2 f(b^\perp, 1) = s(f(b, 1) + f(b^\perp, 1)) = s. \end{aligned}$$

Therefore

$$f(x, 1) = f(x, b) = f(x, b^\perp) = f(z, 1) = s,$$

then $R(z) = \{0, 1\} \subset B_b$, $z(s) = 1$ and moreover $z = E_f(x|B_b)$.

The joint distribution for the observables x, y is given in the 2nd table in Example 2.1. The second and the third columns are $p_{x,y}$ and the fourth and the fifth columns are $p_{y,x}$.

If $R(x) = B_a$ and $x(1) = a$, $x(2) = a^\perp$, then

$$f(x, 1) = f(x, b) = f(x, b^\perp) = 1.6.$$

Let $z := E_f(x|B_b)$. Hence

$$f(x, 1) = f(z, 1) = f(z, b) = f(z, b^\perp) = 1.6.$$

Therefore $E_f(x|B_b)(1.6) = 1$. (In Example 2.1 for any $d \in B_b - \{0\}$ and any $c \in B_a c \asymp_{f(.,1)} d$.)

On the other hand, let $R(y) = B_b$, $y(1) = b$, $y(2) = b^\perp$ and $w := E_f(y|B_a)$. Hence

$$f(y,1) = 1.7 = 0.4w_1 + 0.6w_2$$

$$f(y,a) = 1.8 = w_1, \qquad f(y,a^\perp) = \frac{49}{30} = w_2$$

and so

$$E_f(y|B_a)(1.8) = a, \quad E_f(y|B_a)\left(\frac{49}{30}\right) = a^\perp.$$

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